Inverse Functions

An inverse function "undoes" a function, and is found by interchanging variables (x becomes y, and y becomes x).

Properties of Inverse Functions

- For a function $f(x)$, the inverse notation for the function is $f^{-1}(x)$
- The domain of $f(x)$ is equal to the range of $f^{-1}(x)$
- The range of $f(x)$ is equal to the domain of $f^{-1}(x)$
- A function $f(x)$ must be a one-to-one function in order to have an inverse $f^{-1}(x)$
- $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$

Example 1: Find the inverse of the function $f(x) = 2x^3 + 1$

Step 1: Replace $f(x)$ with y to see as an equation with variables x and y

$$y = 2x^3 + 1$$

Step 2: Interchange x and y

$$x = 2y^3 + 1$$

Step 3: Solve for y

$$x - 1 = 2y^3$$

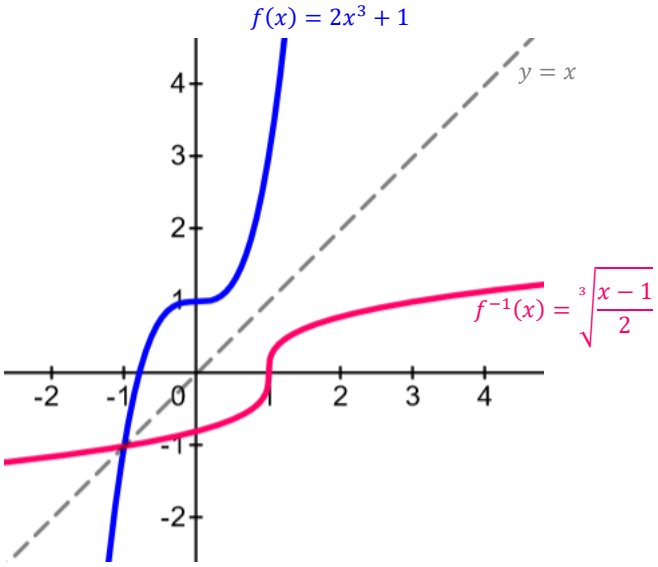
$$\frac{x-1}{2} = y^3 \qquad \text{cube root both sides.}$$

$$\sqrt[3]{\frac{x-1}{2}} = y \longrightarrow y = \sqrt[3]{\frac{x-1}{2}}$$

Step 4: Replace y with $f^{-1}(x)$ to write in inverse notation.

$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$$

Graph to verify:



Note:
The graphs of f and f^{-1} should always be symmetrical reflections along the line $y = x$.

Example 2: Verify that f and g are inverses of each other,

given that $f(x) = \dfrac{x-9}{4}$, $g(x) = 4x + 9$

Step 1: For inverse functions, $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$

So to verify the functions f and g are inverses, show that $f\left(g(x)\right) = x$ and $g\left(f(x)\right) = x$

$$f\left(g(x)\right) = \frac{(4x+9)-9}{4} = \frac{4x}{4} = x$$

$$g\left(f(x)\right) = 4\left(\frac{x-9}{4}\right) + 9 = x - 9 + 9 = x$$

Therefore, the functions f and g are inverses of each other.

Example 3: Assume f is a one-to-one function with domain: $(-\infty, \infty)$

If $f(7) = 4$, find $f^{-1}(4)$

Step 1: For $f(7) = 4$, the input is $x = 7$ and the output is $y = 4$

So for inverse, the input becomes the output, and output becomes the input.

Therefore for $f^{-1}(4)$, when the input is 4, the output is 7

$$f^{-1}(4) = 7$$

Example 4: Use the given table for $y = f(x)$ to complete a table for $y = f^{-1}(x)$

$y = f(x)$

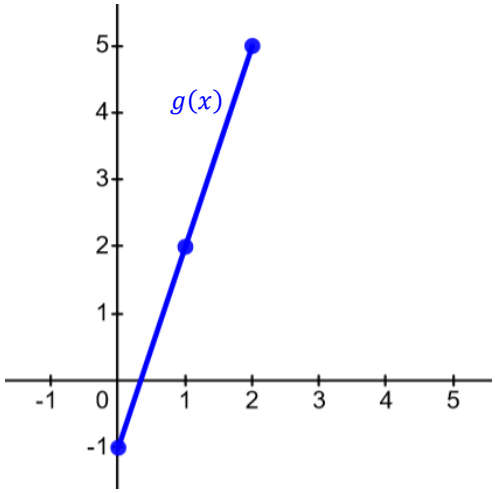
x	y
1	5
-2	0
6	6
1	-3

Step 1:
Interchange the x and y values:

$y = f^{-1}(x)$

x	y
5	1
0	-2
6	6
-3	1

Example 5: Draw the graph of $g^{-1}(x)$ on the same axis as the graph of the function $g(x)$ given below:



To draw the graph of the inverse…

Step 1: Note the points on the given graph of $g(x)$ are: $(0,-1)$ $(1,2)$ $(2,5)$

Step 2: Find the inverse points by interchanging x and y for each point.

The inverse points are: $(-1,0)$ $(2,1)$ $(5,2)$

Step 3: Plot and connect the points to graph the inverse $g^{-1}(x)$