Polynomial Functions

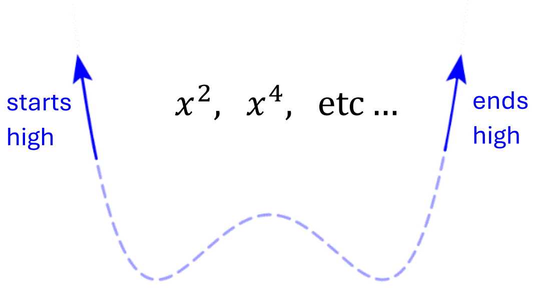
A polynomial function is a function such as linear, quadratic, cubic, that involve only positive-integer powers on the variables.

Examples of polynomial functions:

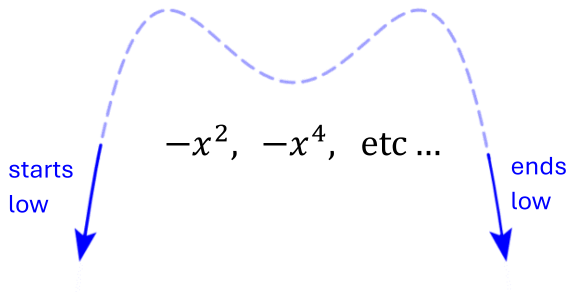
Function	Degree	Leading Term	Leading Coefficient
$f(x) = \frac{1}{2}x^3 - 10x + 8$	3 (cubic)	$\frac{1}{2}x^3$	$\frac{1}{2}$
$y = 5x + 7$	1 (linear)	$5x$	5
$g(x) = 7x - x^6 + 4$	6	$-x^6$	-1
$f(x) = 4x^3(x-1)^2$ $= 4x^5 - 8x^4 + 4x^3$	5	$4x^5$	4

End Behavior

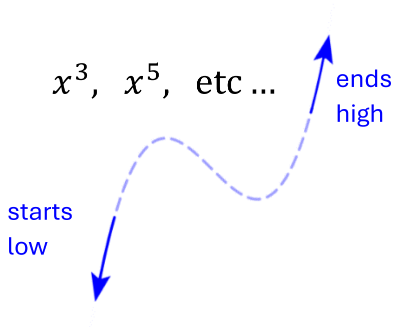
Even degree: the graph starts high, ends high



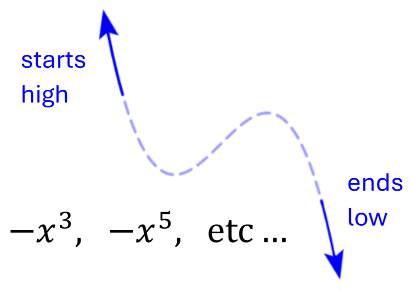
(When leading term is negative, the graph flips upside down)



Odd degree: the graph starts low, ends high

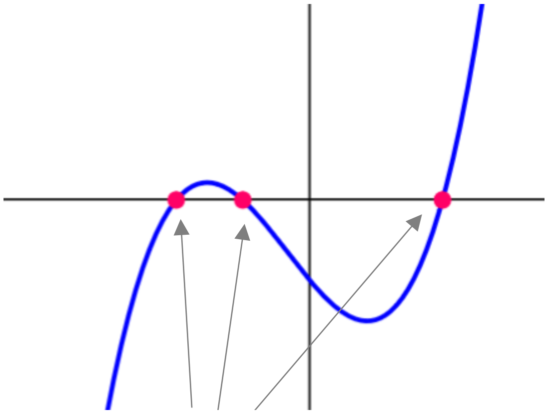


(When leading term is negative, the graph flips upside down)



Zeros of Polynomial Functions

The zeros of a function are the values that make the function equal zero. Zeros are also called roots and x-intercepts.



Zeros (roots, x-intercepts)

Multiplicity – the number of times a factor appears in the factored polynomial (the degree of the factor)

When the multiplicity is odd, the graph crosses through.
When the multiplicity is even, the graph bounces.

Example: Find the real zeros, then graph the function $f(x) = -(x-2)(x+1)^2(x-3)$

Step 1: Determine the end behavior.

Adding up the degrees of each factor, the degree of the polynomial is 4 and the leading term is negative, so the end behavior is "starts low, ends low".

Step 2: Set the function to zero and solve for x to find the zeros.

$$-(x-2)(x+1)^2(x-3) = 0$$

Since the function is already factored, set each factor to zero and solve each for x.

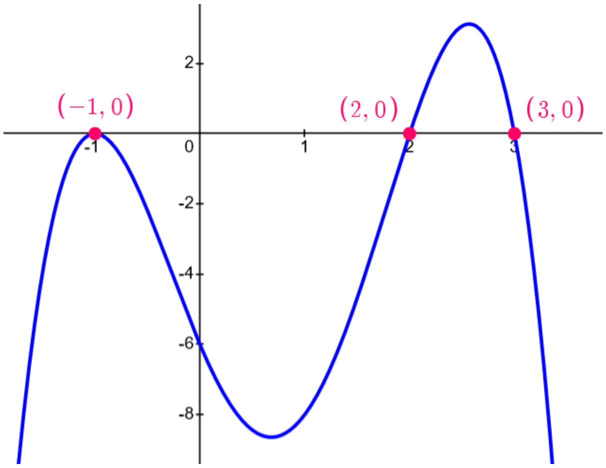
$x - 2 = 0 \quad \Rightarrow \quad x = 2$ This factor has degree of 1 so multiplicity of 1 so graph will cross at $x = 2$

$x + 1 = 0 \quad \Rightarrow \quad x = -1$ This factor has degree of 2 so multiplicity of 2 so graph will bounce at $x = -1$

$x - 3 = 0 \quad \Rightarrow \quad x = 3$ This factor has degree of 1 so multiplicity of 1 so graph will cross at $x = 3$.

So the zeros are:

$x = 2$	multiplicity 1
$x = -1$	multiplicity 2
$x = 3$	multiplicity 1



Rational Zero Theorem

The Rational Zero Theorem finds a list of possible rational zeros of a polynomial.

If a polynomial has integer coefficients,
then every rational zero of the function has the form: $\frac{p}{q}$

Where $p =$ a factor of the constant term and $q =$ a factor of the leading coefficient

Example: List the possible rational zeros of the function $f(x) = 2x^3 - 3x^2 - x + 8$

Step 1: $p = 2$ the coefficient of the leading term,
$q = 8$ the constant term (last term when in descending order).

Step 2: Find all the factors of each number and write in the form $\frac{p}{q}$

$$\frac{p}{q} = \frac{\pm 8,\ \pm 4,\ \pm 2,\ \pm 1}{\pm 2,\ \pm 1}$$

Step 3: Write each top number over each bottom number and simplify.

$$\pm\frac{8}{1},\ \pm\frac{8}{2},\ \pm\frac{4}{1},\ \pm\frac{4}{2},\ \pm\frac{2}{1},\ \pm\frac{2}{2},\ \pm\frac{1}{1},\ \pm\frac{1}{2}$$

The possible rational zeros are: $-1,\ 1,\ -2,\ 2,\ -4,\ 4,\ -8,\ 8,\ -\frac{1}{2},\ \frac{1}{2}$

Factor Theorem

A polynomial $f(x)$ has a factor $(x - c)$ if and only if $f(c) = 0$.

Example: Determine if $(x - 2)$ is a factor of $f(x) = 5x^3 - 10x^2 - 5x + 10$

Step 1: $c = 2$, so plug 2 into x of the function.

$$f(2) = 5(2)^3 - 10(2)^2 - 5(2) + 10 \quad = 0$$

$$f(2) = 0$$

Therefore $(x - 2)$ is a factor of $f(x) = 5x^3 - 10x^2 - 5x + 10$

Descartes Rule of Signs

For a polynomial, the number of real zeros equals the number of sign changes
OR sign changes minus a positive even integer.

Look for sign changes in $f(x)$ and $f(-x)$. If no sign changes, there are no real zeros.

Example: For the function $f(x) = 2x^5 - 3x^2 + x - 4$
List the number of possible positive and negative real zeros using Descartes' Rule of Signs

Step 1: Determine sign changes of $f(x)$

$$f(x) = 2x^5 - 3x^2 + x - 4$$

There are 3 sign changes in $f(x)$.
We can subtract the even integer of 2. $3 - 2 = 1$.

So the function has 3 or 1 positive real zeros.

Step 2: Find $f(-x)$ to determine number of negative real zeros.

$$f(-x) = 2(-x)^5 - 3(-x)^2 + (-x) - 4 \quad \Rightarrow \quad f(-x) = -2x^5 - 3x^2 - x - 4$$

Step 3: Determine sign changes of $f(-x)$

$$f(-x) = -2x^5 - 3x^2 - x - 4$$

There are NO sign changes in $f(-x)$

So the function has no negative real zeros.

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is the value $r = f(c)$

Example: Use the Remainder Theorem to find $f(c)$ for $f(x) = x^3 + 2x^2 - x + 9$ when $c = 3$

Step 1: Use synthetic division to divide

$$\begin{array}{r|rrrr} 3 & 1 & 2 & -1 & 9 \\ & & & & \\ \hline \end{array} \qquad \begin{array}{r|rrrr} 3 & 1 & 2 & -1 & 9 \\ & \downarrow & 3 & 15 & 42 \\ \hline & 1 & 5 & 14 & \boxed{51} \end{array}$$

Remainder

So $r = f(3) = 51$

Check: We can plug 3 into the function to verify we get the same result.

$$f(3) = (3)^3 + 2(3)^2 - (3) + 9 = 51$$

Finding Zeros of Polynomial Functions

Example 1: Find all zeros for the function $f(x) = x^3 + 8x^2 + 13x + 6$

Step 1: Use the Rational Zeros Theorem to find the possible real zeros.

$$\frac{p}{q} = \frac{\pm 6, \quad \pm 3, \quad \pm 2, \quad \pm 1}{\pm 1}$$

So the possible real zeros are: $-1, \ 1, \ -2, \ 2, \ -3, \ 3, \ -6, \ 6$

Step 2: Use synthetic division to divide the polynomial by the possible zeros, until one gets a remainder of zero.

$$\begin{array}{r|rrrr} -1 & 1 & 8 & 13 & 6 \\ & \downarrow & -1 & -7 & -6 \\ \hline & 1 & 7 & 6 & \boxed{0} \end{array}$$

Remainder

-1 gets a remainder of zero, so $x = -1$ is a zero of the polynomial.

The result of synthetic division is always one less degree than the original polynomial, so

$1 \quad 7 \quad 6 \ = x^2 + 7x + 6$

Step 3: Since the result is quadratic, use factoring or quadratic formula to find remaining zeros.

Factoring, $x^2 + 7x + 6 = 0 \quad \Rightarrow \quad (x+6)(x+1) = 0 \quad \Rightarrow \quad x = -6, \ x = -1$

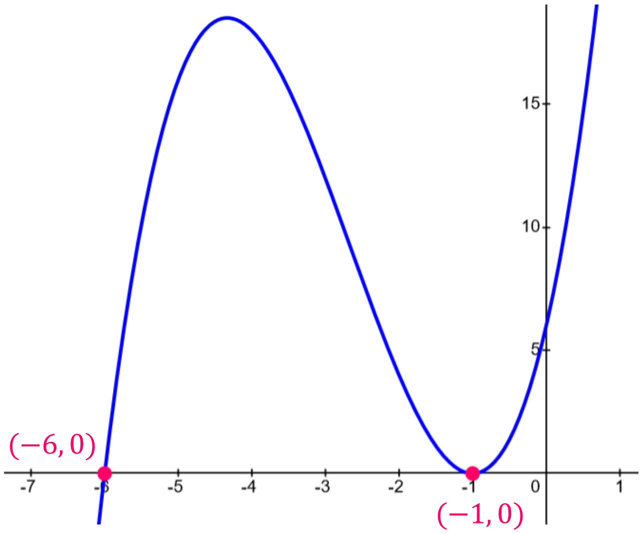
So the remaining zeros are -1 and -6

So the zeros of the function are -1 (multiplicity 2) and -6.

Note: -1 has a multiplicity of 2 since it was found twice to be a zero.

Factored form of the function: $f(x) = (x+6)(x+1)^2$

Graph to confirm:



Finding Zeros of Polynomial Functions

Example 2: Find all zeros for the function $f(x) = 2x^4 + 3x^3 - 16x^2 + 15x - 4$

Step 1: Use the Rational Zeros Theorem to find the possible real zeros.

$$\frac{p}{q} = \frac{\pm 4, \quad \pm 2, \quad \pm 1}{\pm 2, \quad \pm 1}$$

So the possible real zeros are: $-1,\ 1, -2,\ 2, -4,\ 4,\ -\frac{1}{2},\ \frac{1}{2}$

Step 2: Use synthetic division to divide the polynomial by the possible zeros, until one gets a remainder of zero.

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & -16 & 15 & -4 \\ & \downarrow & 2 & 5 & -11 & 4 \\ \hline & 2 & 5 & -11 & 4 & \boxed{0} \end{array} \longleftarrow \text{Remainder}$$

1 gets a remainder of zero, so $x = 1$ is a zero of the polynomial.

Step 3: Use the result to divide again to find another zero.

$$\begin{array}{r|rrrr} -4 & 2 & 5 & -11 & 4 \\ & \downarrow & -8 & 12 & -4 \\ \hline & 2 & -3 & 1 & \boxed{0} \end{array} \longleftarrow \text{Remainder}$$

-4 gets a remainder of zero, so $x = -4$ is a zero of the polynomial.

Step 4: Use factoring or quadratic formula to find remaining zeros.

Factoring, $2x^2 - 3x + 1 = 0 \quad \Rightarrow \quad (2x - 1)(x - 1) = 0 \quad \Rightarrow \quad x = \frac{1}{2},\ x = 1$

So the remaining zeros are $-1/2$ and 1

So the zeros of the function are 1 (multiplicity 2), -4 and $-1/2$.

Note: 1 has a multiplicity of 2 since it was found twice to be a zero.

Factored form of the function: $f(x) = (x - 1)^2(x + 4)(2x - 1)$