Rational Expressions

A rational expression is a quotient of two polynomials (polynomials can include monomials).

Examples of rational expressions:	$\frac{5}{7}$	$\frac{1}{x}$	$\frac{2}{x-4}$	$\frac{x(x-1)}{x^2-4}$	$\frac{x^2+3xy+y^2}{2x^5-6xy+7y^2}$

Rational expressions are undefined when the denominator is equal to zero, since dividing by zero is undefined.

Example: Find the domain of the rational expression $\frac{3x-4}{x-5}$

Step 1: Set the denominator to zero and solve for x

$x - 5 = 0$

$x = 5$, therefore the rational expression is undefined when $x = 5$

So the domain is all real numbers except 5, which in interval form is $(-\infty, 5) \cup (5, \infty)$

Simplifying Rational Expressions

Example 1: Simplify the rational expression $\frac{x+1}{x^2-4x-5}$

Step 1: Factor the top and bottom if possible

$$\frac{x+1}{x^2-4x-5} = \frac{x+1}{(x-5)(x+1)}$$

Step 2: Cancel like terms

$$\frac{\cancel{x+1}}{(x-5)\cancel{(x+1)}} = \frac{1}{x-5}$$

Example 2: Simplify $\frac{y-9}{9-y}$

Note: $9 - y = -1(-9+y) = -1(y-9)$

So, $\frac{y-9}{9-y} = \frac{\cancel{y-9}}{-\cancel{(y-9)}} = \frac{1}{-1} = -1$

Multiplying Rational Expressions

Example: Multiply and simplify the rational expressions $\frac{x-5}{x+4} \cdot \frac{8x+32}{7x-35}$

Step 1: Factor the top and bottom if possible

$$\frac{x-5}{x+4} \cdot \frac{8x+32}{7x-35} = \frac{x-5}{x+4} \cdot \frac{8(x+4)}{7(x-5)}$$

Step 2: Cancel like terms

$$= \frac{\cancel{x-5}}{\cancel{x+4}} \cdot \frac{8\cancel{(x+4)}}{7\cancel{(x-5)}} = \frac{8}{7}$$

Dividing Rational Expressions

Example: Divide and simplify the rational expressions $\frac{x+3}{x-3} \div \frac{x^2-x-2}{x^2-9}$

Step 1: Change to multiplication by flipping the 2nd expression (multiply by the reciprocal)

$$\frac{x+3}{x-3} \div \frac{x^2-x-2}{x^2-9} \quad = \frac{x+3}{x-3} \cdot \frac{x^2-9}{x^2-x-2}$$

Step 2: Factor, then cancel like terms

$$= \frac{x+3}{\cancel{x-3}} \cdot \frac{(\cancel{x-3})(x+3)}{(x-2)(x+1)} \quad = \frac{x+3}{1} \cdot \frac{x+3}{(x-2)(x+1)} \quad = \frac{(x+3)^2}{(x-2)(x+1)}$$

Adding and Subtracting Rational Expressions

If denominators are the same, combine like terms on the numerators.

Example: Add $\frac{7x+20}{2x+3} + \frac{9x+4}{2x+3}$

Step 1: Denominators are the same, so add like terms on the top

$$\frac{7x+20}{2x+3} + \frac{9x+4}{2x+3} \quad = \frac{7x+20+9x+4}{2x+3} \quad = \frac{16x+24}{2x+3}$$

Step 2: Factor, then cancel like terms

$$= \frac{16x+24}{2x+3} \quad = \frac{8(\cancel{2x+3})}{\cancel{2x+3}} \quad = \frac{8}{1} \quad = 8$$

If unlike denominators, find lowest common denominator (LCD) first.

Example: Subtract $\frac{7}{x-4} - \frac{2}{x+7}$

Step 1: Since unlike denominators, find the least common denominator

$$\text{LCD} = (x-4)(x+7)$$

Step 2: Multiply each side by what is needed to make the LCD

$$\frac{(x+7)}{(x+7)} \cdot \frac{7}{x-4} - \frac{2}{x+7} \cdot \frac{(x-4)}{(x-4)} \quad = \frac{7(x+7)}{(x+7)(x-4)} - \frac{2(x-4)}{(x+7)(x-4)}$$

Step 3: Write all over the same denominator and simplify numerator

$$= \frac{7(x+7)-2(x-4)}{(x+7)(x-4)} \quad = \frac{7x+49-2x+8}{(x+7)(x-4)} \quad = \frac{5x+57}{(x+7)(x-4)}$$

Complex Rational Expressions (Complex Fractions)

A complex rational has rational expression in its numerator, denominator, or both.

There are 2 methods generally used to simply complex rational expressions.

Method 1: Find the LCD for the all denominators in the complex rational expression

Example: Simplify $\dfrac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$

Step 1: Find the LCD for all denominators

$\text{LCD} = 4x^2$

Step 2: Multiply top and bottom by the LCD

$$\frac{\left(\frac{1}{2}+\frac{1}{x}\right)}{\left(\frac{1}{4}-\frac{1}{x^2}\right)}\cdot\frac{\left(\frac{4x^2}{1}\right)}{\left(\frac{4x^2}{1}\right)} = \frac{\frac{4x^2}{2}+\frac{4x^2}{x}}{\frac{4x^2}{4}-\frac{4x^2}{x^2}} = \frac{2x^2+4x}{x^2-4}$$

Step 3: Factor and simplify

$$= \frac{2x^2+4x}{x^2-4} = \frac{2x(x+2)}{(x-2)(x+2)} = \frac{2x}{x-2}$$

Method 2: Simplify the numerator and denominator separately

Example: Simplify $\dfrac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$

Step 1: Find the LCD for the numerator and denominator separately

For the numerator, the LCD $= 2x$ For the denominator, the LCD $= 4x^2$

Step 2: Multiply to get common denominator and then add or subtract fractions

$$\frac{\frac{x}{x}\cdot\frac{1}{2}+\frac{1}{x}\cdot\frac{2}{2}}{\frac{x^2}{x^2}\cdot\frac{1}{4}-\frac{1}{x^2}\cdot\frac{4}{4}} = \frac{\frac{x}{2x}+\frac{2}{2x}}{\frac{x^2}{4x^2}-\frac{4}{4x^2}} = \frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}}$$

Step 3: Change division of fractions to multiplication of the reciprocal

$$\frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}} = \frac{x+2}{2x}\cdot\frac{4x^2}{x^2-4}$$

Step 4: Factor and simplify

$$\frac{x+2}{2x}\cdot\frac{4x^2}{x^2-4} = \frac{x+2}{2x}\cdot\frac{\overset{2x}{4x^2}}{(x-2)(x+2)} = \frac{2x}{x-2}$$

We see the same answer as method 1

Solving Rational Equations

A rational equation is an equation with at least one rational expression.

Example: Solve the equation $\frac{12}{x-5} - \frac{8}{x-4} = \frac{8}{x^2 - 9x + 20}$

Step 1: Factor the denominators if possible

$$\frac{12}{x-5} - \frac{8}{x-4} = \frac{8}{(x-5)(x-4)}$$

Step 2: Find the LCD

The LCD is $(x-5)(x-4)$

Step 3: Multiply the equation by the LCD to cancel the fractions

$$\frac{(x-5)(x-4)}{1}\left(\frac{12}{x-5} - \frac{8}{x-4} = \frac{8}{(x-5)(x-4)}\right)$$

$$\left(\frac{12(x-5)(x-4)}{x-5} - \frac{8(x-5)(x-4)}{x-4} = \frac{8(x-5)(x-4)}{(x-5)(x-4)}\right)$$

$$12(x-4) - 8(x-5) = 8$$

Step 4: Solve for x

$$12(x-4) - 8(x-5) = 8 \quad \Rightarrow \quad 12x - 48 - 8x + 40 = 8$$

$$4x - 8 = 8 \quad \Rightarrow \quad 4x = 16 \quad \Rightarrow \quad x = 4$$

Step 5: Test the answer to see if it makes the equation undefined

Plugging 4 in for x into the denominator $x - 4$ results in 0, and dividing by zero is undefined.

Therefore $x = 4$ is not a solution.

So there is NO SOLUTION for this equation.