Rational Functions

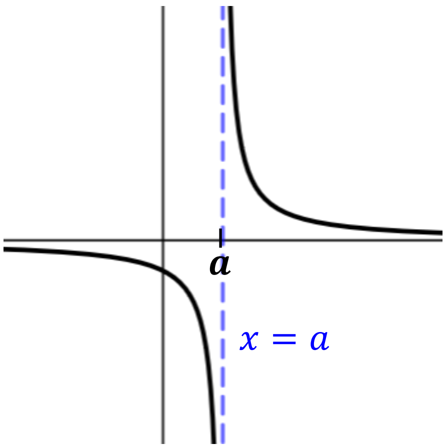
A rational function is a function that is a quotient of two polynomials (polynomials can include monomials).

Examples of rational functions:	$f(x) = \frac{1}{x}$	$y = \frac{2x-1}{x+3}$	$g(x) = \frac{x-3}{5x^2+31x+6}$

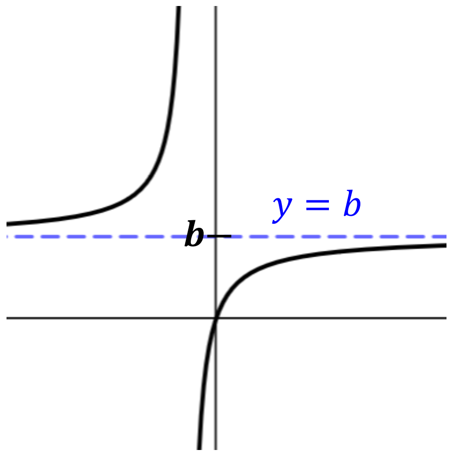
Asymptotes

Asymptote – a line that a curve approaches, as either the x or y coordinate approaches infinity or negative infinity.
(Asymptotes often show up in graphs of rational functions)

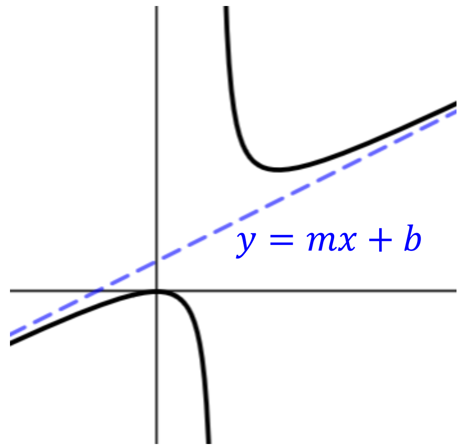
Vertical Asymptote



Horizontal Asymptote



Oblique (Slant) Asymptote



Vertical Asymptotes

To find vertical asymptotes – factor and cancel if possible, then set denominator to zero and solve for x.

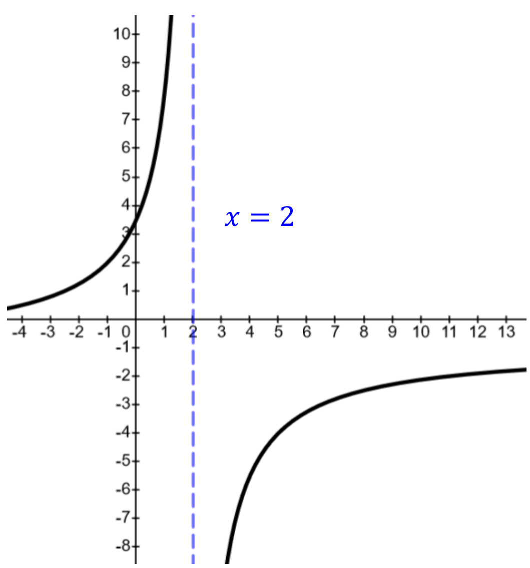
Example: Find vertical asymptotes for the function $f(x) = \frac{x+7}{2-x}$

Step 1: Nothing can be factored or cancelled, so set denominator to zero and solve for x

$$2 - x = 0 \quad \Rightarrow \quad -x = -2 \quad \Rightarrow \quad x = 2$$

So the vertical asymptote is $x = 2$

Graph to verify:



Horizontal Asymptotes

Finding horizontal asymptotes:

- If the degree of the numerator is 1 degree larger than the degree of the denominator, the horizontal asymptote does not exist (there is an oblique/slant asymptote instead).
- If degree of the numerator is less than the degree of the denominator, the horizontal asymptote is zero.
- If the degree of the numerator is the same as the degree of the denominator, the horizontal asymptote is the ratio of the leading coefficients.

Examples: Find the horizontal asymptote for the following functions:

$$f(x) = \frac{6x^3 + 4}{5x^2 - 3x}$$ Degree is higher on top, so the horizontal asymptote does not exist.

$$f(x) = \frac{6x^2 + 4}{5x^3 - 3x}$$ Degree is higher on the bottom, so the horizontal asymptote is $y = 0$

$$f(x) = \frac{6x^2 + 4}{5x^2 - 3x}$$ Degrees are the same on top and bottom, so the horizontal asymptote is $y = \frac{6}{5}$

Oblique (Slant) Asymptotes

To find oblique asymptotes, divide numerator by the denominator to get a linear equation.

Example: Find the horizontal or oblique asymptote of the function $f(x) = \frac{x^2 + 4x - 1}{x + 3}$

Step 1: Note that the numerator is 1 degree larger than the denominator, so there is no horizontal asymptote, instead there is an oblique asymptote.

Step 2: Divide using long division, or synthetic division

Dividing using synthetic division, $x + 3 = 0 \quad \Rightarrow \quad x = -3,$ so divide by -3

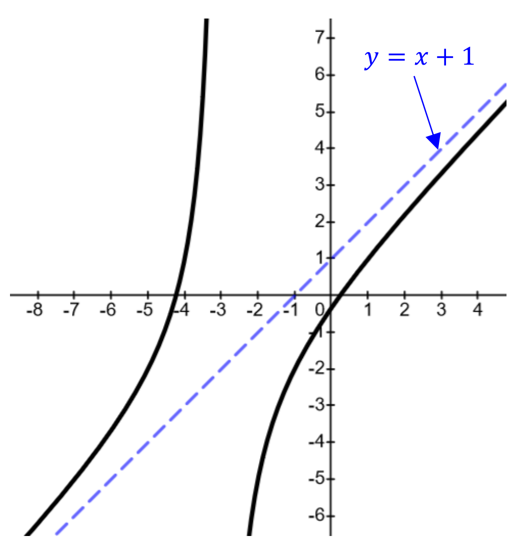
-3	1	4	-1

-3	1	4	-1
		-3	-3
	1	1	-4

← Ignore remainder

Write as equation
$y = 1x + 1$

So the oblique asymptote is $y = x + 1$



Example: Find all asymptotes, x and y-intercepts and domain of the function

$$f(x) = \frac{x+3}{2x^2 - 5x - 3}$$

Step 1: Factor and cancel if possible

$$\frac{x+3}{2x^2 - 5x - 3} = \frac{x+3}{(2x+1)(x-3)} \quad \text{(Nothing cancels out)}$$

Step 2: Find vertical asymptotes by setting denominator to zero and solving for x

$(2x+1)(x-3) = 0$ Solving each factor for x gets $x = -\frac{1}{2}$, $x = 3$ Which are the vertical asymptotes.

Step 3: Find horizontal asymptotes by looking at the degree of the numerator and denominator

$\frac{x+3}{2x^2 - 5x - 3}$ The degree is higher on the bottom, so the horizontal asymptote is $y = 0$

Step 4: Find the y-intercepts by plugging zero in for x in the function

$$\frac{x+3}{2x^2 - 5x - 3} = \frac{(0)+3}{2(0)^2 - 5(0) - 3} = \frac{3}{-3} = -1$$ So the y-intercept is $(0, -1)$

Step 5: Find the x-intercepts by setting the function equal to zero

$0 = \frac{x+3}{2x^2 - 5x - 3}$ Multiply both sides by denominator to cancel fractions

$$\left(0 = \frac{x+3}{2x^2 - 5x - 3}\right) \cdot \frac{2x^2 - 5x - 3}{1} \Rightarrow 0 = x + 3 \Rightarrow x = -3$$

So the x-intercept is $(-3, 0)$

Step 6: Graph