# Chain Rule

The Chain Rule is used to find derivatives of composite functions.
To find the derivative of a composite function $f \circ g$, you can multiply the derivatives of $f$ and $g$.

**Chain Rule:**

For some composite function: $y = f(g(x))$

The derivative is: $y' = f'(g(x)) \cdot g'(x)$

**Alternate Notation for Chain Rule:**

If we set $u = g(x)$, then $y = f(g(x))$ can be written as: $y = f(u)$

so the derivative can be written as: $y' = f'(u) \cdot u'$

or also as: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

**General Power Rule:**

The General Power Rule is a special case of the Chain Rule,
which combines the Chain Rule and Power Rule.

For a function in the form: $y = (g(x))^n$

The derivative is: $y' = n \cdot (g(x))^{n-1} \cdot g'(x)$

**Alternate Notation for the General Power Rule:**

If we set $u = g(x)$, then $y = (g(x))^n$ can be written as: $y = u^n$

so the derivative can be written as: $y' = nu^{n-1} \cdot u'$

# Chain Rule

**Example 1:** Find the derivative of the function $y = -7(8x^2+3)^6$

Method 1:

Step 1: Decompose the function to identify the "outer" function $f$ and "inner" function $g$

The outer function is $f(x) = -7x^6$ and the inner function is $g(x) = 8x^2 + 3$

Then find their derivatives: $f'(x) = -42x^5$ and $g'(x) = 16x$

Step 2: Plug in to the Chain Rule.

$$
\begin{aligned}
y' = f'(g(x)) \cdot g'(x) &= -42(8x^2+3)^5 \cdot (16x) \\
&= -672x(8x^2+3)^5
\end{aligned}
$$

Method 2:

Since $y = -7(8x^2+3)^6$ is in the form of $y = (g(x))^n$, use the General Power Rule $\frac{d}{dx}(u^n) = nu^{n-1} \cdot u'$

$$
\begin{aligned}
\frac{d}{dx}(-7(8x^2+3)^6) &= -7(6)(8x^2+3)^{6-1} \cdot (16x) \\
&= -42(8x^2+3)^5 \cdot (16x) \\
&= -672x(8x^2+3)^5
\end{aligned}
$$

**Example 2:** Find the derivative of the function $h(x) = e^{-x^3+4x}$

Method 1:

Step 1: Decompose the function to identify the "outer" function $f$ and the "inner" function $g$

The outer function is $f(x) = e^x$ and the inner function is $g(x) = -x^3 + 4x$

Then find their derivatives: $f'(x) = e^x$ and $g'(x) = -3x^2 + 4$

Step 2: Plug in to the Chain Rule.

$$
\begin{aligned}
h'(x) = f'(g(x)) \cdot g'(x) &= e^{-x^3+4x} \cdot (-3x^2+4) \\
&= (-3x^2+4)e^{-3x+4}
\end{aligned}
$$

Method 2:

Another way to show this is using the Chain Rule for the Natural Exponential function: $\frac{d}{dx}(e^u) = e^u \cdot u'$

$$
\begin{aligned}
\frac{d}{dx}\left(e^{-x^3+4x}\right) &= e^{-x^3+4x} \cdot (-3x^2+4) \\
&= (-3x^2+4)e^{-3x+4}
\end{aligned}
$$

# Chain Rule

**Example 3:** Find the derivative of the function $F(x) = \ln(2x^2 + 1)$

Method 1:

Step 1: Decompose the function to identify the "outer" function $f$ and the "inner" function $g$

The outer function is $f(x) = \ln x$ and the inner function is $g(x) = 2x^2 + 1$

Then find their derivatives: $f'(x) = \frac{1}{x}$ and $g'(x) = 4x$

Step 2: Plug in to the Chain Rule.

$$F'(x) = f'\big(g(x)\big) \cdot g'(x) = \frac{1}{2x^2 + 1} \cdot (4x) = \frac{4x}{2x^2 + 1}$$

Method 2:

Another way to show this is using the Chain Rule for the Natural Logarithmic function: $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot u' = \frac{u'}{u}$

$$\frac{d}{dx}(\ln(2x^2 + 1)) = \frac{1}{2x^2 + 1} \cdot (4x) = \frac{4x}{2x^2 + 1}$$

**Example 4:** Find the derivative of the function $y = \sin(\pi x)$

Method 1:

Step 1: Decompose the function to identify the "outer" function $f$ and the "inner" function $g$

The outer function is $f(x) = \sin x$ and the inner function is $g(x) = \pi x$

Then find their derivatives: $f'(x) = \cos x$ and $g'(x) = \pi$

Step 2: Plug in to the Chain Rule.

$$y' = f'\big(g(x)\big) \cdot g'(x) = \cos(\pi x) \cdot \pi = \pi \cos(\pi x)$$

Method 2:

Another way to show this is using the Chain Rule for the Sine function: $\frac{d}{dx}(\sin u) = \cos(u) \cdot u'$

$$\frac{d}{dx}(\sin(\pi x)) = \cos(\pi x) \cdot \pi = \pi \cos(\pi x)$$