# Integration by Substitution

*u-substitution* is used often used when the integral does not fit the form of a known integral rule.

These are some examples of the types of integrals that would use u-substitution:

$\int (3x-2)^7 dx$ $\quad u = 3x - 2$ $\quad$ ($u$ is what is in parenthesis to a power)

$\int 16x(8x^2+1)^2 dx$ $\quad u = 8x^2 + 1$ $\quad$ ($u$ is what is in parenthesis to a power)

$\int t^3\sqrt{2t^4+3}\, dt$ $\quad u = 2x^4 + 3$ $\quad$ ($u$ is what is in the root)

$\int xe^{x^2-9} dx$ $\quad u = x^2 - 9$ $\quad$ ($u$ is what is in the exponent)

$\int \frac{x^3}{(1+x^4)^2} dx$ $\quad u = 1 + x^4$ $\quad$ ($u$ is what is in the denominator)

$\int \frac{\sin(\ln x)}{x} dx$ $\quad u = \ln x$ $\quad$ ($u$ is what is inside a trig function)

**Example 1:** Evaluate the indefinite integral $\int (3x-2)^7 dx$

Step 1: Define $u$ and take the derivative

$$\text{Let } u = 3x - 2 \qquad \frac{du}{dx} = 3 \qquad du = 3dx$$

Step 2: Solve for $dx$

$$du = 3dx \quad \Rightarrow \quad \frac{du}{3} = dx \quad \Rightarrow \quad \frac{1}{3}du = dx$$

Step 3: Plug in $u$ and $\frac{1}{3}du$ and write the integral in terms of $u$

$$\int (3x-2)^7 dx \quad = \int u^7 \cdot \frac{1}{3} du \quad = \frac{1}{3}\int u^7 du$$

Step 4: Integrate

$$\frac{1}{3}\int u^7 du \quad = \frac{1}{3} \cdot \frac{u^8}{8} + C \quad = \frac{1}{24}u^8 + C$$

Step 5: Plug back in for $u$ to write in terms of $x$

$$\frac{1}{24}u^8 + C \quad = \frac{1}{24}(3x-2)^8 + C$$

**Example 2:** Evaluate the indefinite integral $\int 16x(8x^2+1)^2dx$

Step 1: Define $u$ and take the derivative

Let $u = 8x^2 + 1$ $\quad \frac{du}{dx} = 16x \quad du = 16xdx$ (note that $16x$ is in the original integral)

Step 2: Plug in $u$ and $du$ and write the integral in terms of $u$

$$\int (8x^2+1)^2(16x)dx \quad = \int u^2du$$

Step 3: Integrate

$$\int u^2du \quad = \frac{u^3}{3} + C \quad = \frac{1}{3}u^3 + C$$

Step 4: Plug back in for $u$ to write in terms of $x$

$$\frac{1}{3}u^3 + C \quad = \frac{1}{3}(8x^2+1)^3 + C$$

**Example 3:** Evaluate the indefinite integral $\int t^3\sqrt{2t^4+3}\,dt$

Step 1: Define $u$ and take the derivative

Let $u = 2t^4 + 3$ $\quad \frac{du}{dt} = 8t^3 \quad du = 8t^3dt$ (note that $t^3$ is in the original integral)

Step 2: Solve for $t^3dt$

$$du = 8t^3dt \quad \Rightarrow \quad \frac{du}{8} = t^3dt \quad \Rightarrow \quad \frac{1}{8}du = t^3dt$$

Step 3: Plug in $u$ and $\frac{1}{8}du$ and write the integral in terms of $u$

$$\int \sqrt{2t^4+3}\,(t^3)dt \quad = \int \sqrt{u}\cdot\frac{1}{8}du \quad = \frac{1}{8}\int u^{1/2}du$$

Step 4: Integrate

$$\frac{1}{8}\int u^{1/2}du \quad = \frac{1}{8}\cdot\frac{u^{3/2}}{\frac{3}{2}} + C \quad = \frac{1}{8}\cdot\frac{2}{3}u^{3/2} + C \quad = \frac{1}{12}u^{3/2} + C$$

Step 5: Plug back in for $u$ to write in terms of $t$

$$\frac{1}{12}u^{3/2} + C \quad = \frac{1}{12}(2t^4+3)^{3/2} + C$$

**Example 4:** Evaluate the indefinite integral $\int xe^{x^2-9}dx$

Step 1: Define $u$ and take the derivative

Let $u = x^2 - 9$ $\quad \frac{du}{dx} = 2x \quad du = 2xdx$ (note that $x$ is in the original integral)

Step 2: Solve for $xdx$

$$du = 2xdx \Rightarrow \frac{du}{2} = xdx \Rightarrow \frac{1}{2}du = xdx$$

Step 3: Plug in $u$ and $\frac{1}{2}du$ and write the integral in terms of $u$

$$\int e^{x^2-9}(x)dx = \int e^u \cdot \frac{1}{2}du = \frac{1}{2}\int e^u du$$

Step 4: Integrate

$$\frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$$

Step 5: Plug back in for $u$ to write in terms of $x$

$$\frac{1}{2}e^u + C = \frac{1}{2}e^{x^2-9} + C$$

**Example 5:** Evaluate the indefinite integral $\int \frac{x^3}{(1+x^4)^2}dx$

Step 1: Define $u$ and take the derivative

Let $u = 1 + x^4$ $\quad \frac{du}{dx} = 4x^3 \quad du = 4x^3dx$ (note that $x^3$ is in the original integral)

Step 2: Solve for $x^3dx$

$$du = 4x^3dx \Rightarrow \frac{du}{4} = x^3dx \Rightarrow \frac{1}{4}du = x^3dx$$

Step 3: Plug in $u$ and $\frac{1}{4}du$ and write the integral in terms of $u$

$$\int \frac{x^3}{(1+x^4)^2}dx = \int \frac{1}{(1+x^4)^2} \cdot x^3dx = \int \frac{1}{u^2} \cdot \frac{1}{4}du = \frac{1}{4}\int u^{-2}du$$

Step 4: Integrate

$$\frac{1}{4}\int u^{-2}du = \frac{1}{4} \cdot \frac{u^{-1}}{-1} + C = -\frac{1}{4}u^{-1} + C = -\frac{1}{4u} + C$$

Step 5: Plug back in for $u$ to write in terms of $x$

$$-\frac{1}{4u} + C = -\frac{1}{4(1+x^4)} + C$$

**Example 6:** Evaluate the indefinite integral $\int \frac{\sin(\ln x)}{x} dx$

Step 1: Define $u$ and take the derivative

Let $u = \ln x$ $\qquad \frac{du}{dx} = \frac{1}{x}$ $\qquad du = \frac{1}{x} dx$ (note that $\frac{1}{x}$ is in the original integral)

Step 2: Plug in $u$ and $du$ and write the integral in terms of $u$

$$\int \frac{\sin(\ln x)}{x} dx \quad = \int \sin(\ln x) \cdot \frac{1}{x} dx \quad = \int \sin u \, du$$

Step 3: Integrate

$$\int \sin u \, du \quad = -\cos u + C$$

Step 4: Plug back in for $u$ to write in terms of $x$

$$-\cos u + C \quad = -\cos(\ln x) + C$$