Limits

Limit – the value that $f(x)$ gets closer to as x approaches some value from both sides (2-sided limit).

$$\lim_{x \to a} f(x) = L$$

This notation in words is saying "the limit of $f(x)$ as x approaches a, equals L."

One sided limits:

Left-handed limit: $\lim_{x \to a^-} f(x)$

- the value that $f(x)$ gets closer to as x approaches some value from the left side.

Right-handed limit: $\lim_{x \to a^+} f(x)$

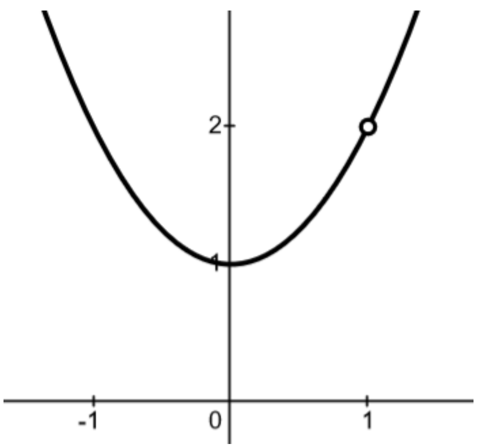
- the value that $f(x)$ gets closer to as x approaches some value from the right side.

The ordinary (2-sided) limit exist if and only if the left- and right-handed limits are the same:

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Example 1: Use the graph of the function $f(x)$ to find the following limits :

a) $\lim_{x \to 1^-} f(x)$ b) $\lim_{x \to 1^+} f(x)$ c) $\lim_{x \to 1} f(x)$



a) Tracing the graph from the left side approaching the x value of 1, the y value approaches 2, so $\lim_{x \to 1^-} f(x) = 2$

b) Tracing the graph from the right side approaching the x value of 1, the y value approaches 2, so $\lim_{x \to 1^+} f(x) = 2$

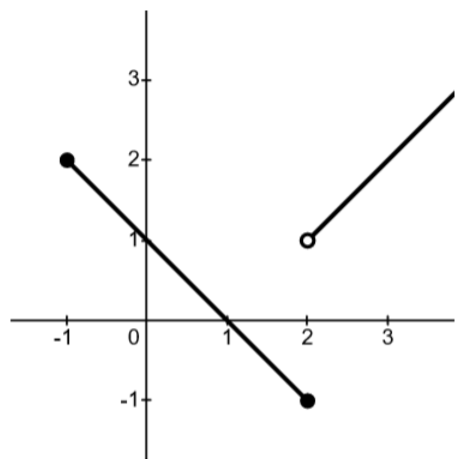
c) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2$

The left and right-hand limits are equal,

Therefore, $\lim_{x \to 1} f(x) = 2$

Example 2: Use the graph of the function $g(x)$ to find the following limits:

a) $\lim_{x \to 2^-} g(x)$ b) $\lim_{x \to 2^+} g(x)$ c) $\lim_{x \to 2} g(x)$



a) Tracing the graph from the left side approaching the x value of 2, the y value approaches -1, so $\lim_{x \to 2^-} g(x) = -1$

b) Tracing the graph from the right side approaching the x value of 2, the y value approaches 1, so $\lim_{x \to 2^+} g(x) = 1$

c) $\lim_{x \to 2^-} g(x) \neq \lim_{x \to 2^+} g(x)$

The left and right-hand limits are NOT equal,

Therefore, $\lim_{x \to 2} g(x)$ does not exist

Estimating Limits Numerically

Example: Use a table to estimate the following limit:

$$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6}$$

Step 1: Plug in x values approaching 2 from the left and right side

for $f(x) = \dfrac{x^2-4}{x^2+x-6}$

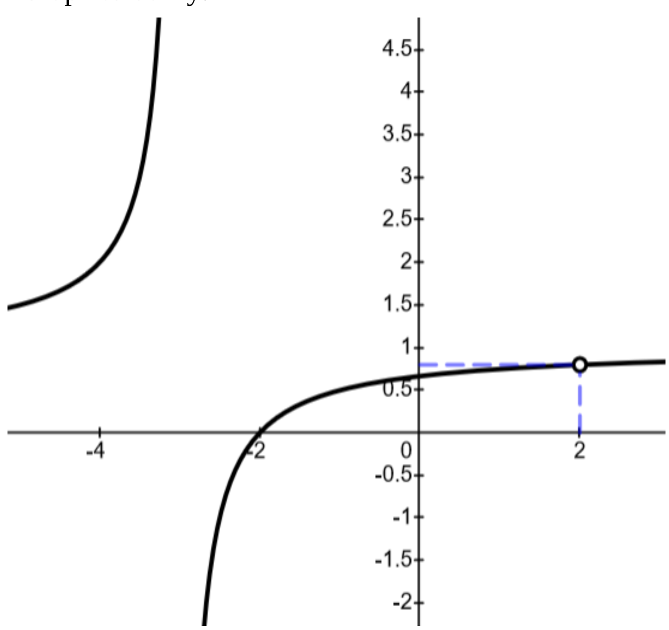
x	$f(x)$
1.9	
1.99	
1.999	
1.9999	
2	
2.0001	
2.001	
2.01	
2.1	

x	$f(x)$
1.9	.79591837
1.99	.7995992
1.999	.79995999
1.9999	.799996
2	undefined
2.0001	.800004
2.001	.80003999
2.01	.8003992
2.1	.80392157

Both sides are approaching .8

so we can estimate that: $\lim_{x\to 2} \dfrac{x^2-4}{x^2+x-6} = .8$

Graph to verify:



Evaluating Limits

Methods to try for finding limits:

- Plug the value in for x if possible
- If you get undefined, try factoring and simplifying
- If square roots, try multiplying by conjugate
- If x goes to infinity, determine horizontal asymptote
- If indeterminate form, try L'Hôpital's Rule
- Squeeze Theorem
- Use a table of values (numerical approach)

Example 1: Find the limit: $\lim_{x\to 2}(5x^3 - 4x)$

Step 1: Plug 2 in for x

$$5(2)^3 - 4(2) = 32 \qquad \text{Therefore: } \lim_{x\to 2}(5x^3 - 4x) = 32$$

Example 2: Find the limit: $\lim_{x\to -2}\dfrac{x^2 - x - 6}{x^2 + 3x + 2}$

Step 1: Plugging in -2 gets zero in the denominator, which makes the expression undefined, so try factoring

$$\frac{x^2 - x - 6}{x^2 + 3x + 2} = \frac{(x-3)(x+2)}{(x+2)(x+1)}$$ Cancel the $x + 2$ on the top and bottom, then plug in -2

$$= \frac{(x-3)\cancel{(x+2)}}{\cancel{(x+2)}(x+1)} = \frac{x-3}{x+1} = \frac{(-2)-3}{(-2)+1} = \frac{-5}{-1} = 5$$

$$\text{Therefore: } \lim_{x\to -2}\frac{x^2 - x - 6}{x^2 + 3x + 2} = 5$$

Example 3: Find the limit: $\lim_{x\to 4}\dfrac{\sqrt{x} - 2}{x - 4}$

Step 1: Plugging in 4 makes the expression 0/0 which is indeterminate form.
Since the expression has a square root, try multiplying by the conjugate.

$$\frac{\sqrt{x}-2}{x-4}\cdot\frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$$ Multiply the binomials on the top: $(\sqrt{x}-2)(\sqrt{x}+2) = x - 4$

$$= \frac{x-4}{(x-4)(\sqrt{x}+2)}$$ Cancel the $x - 4$ on the top and bottom, then plug in 4

$$= \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{(4)}+2} = \frac{1}{4}$$

$$\text{Therefore: } \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \frac{1}{4}$$

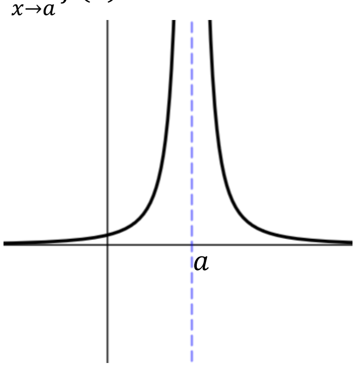
Infinite Limits

Infinite Limits are when $f(x)$ increases without bound to infinity, or decreases without bound to negative infinity.

Since infinity does not represent a real number, technically the limit does not exist,
however, we can describe the behavior of the function saying the limit *approaches* infinity or negative infinity.

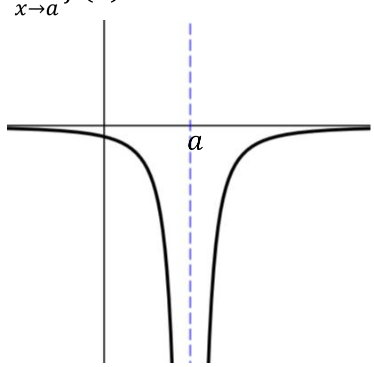
These can be one or two-sided limits, and are located at a vertical asymptote $x = a$

$\lim_{x \to a} f(x) = \infty$



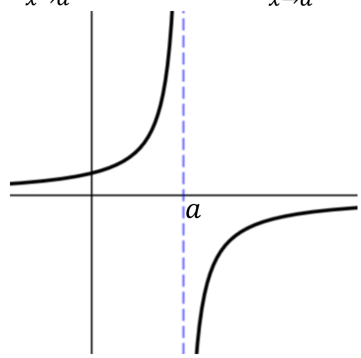
Limit from each side is infinity

$\lim_{x \to a} f(x) = -\infty$



Limit from each side is negative infinity

$\lim_{x \to a^-} f(x) = \infty$ $\lim_{x \to a^+} f(x) = -\infty$



Limit from the left is infinity
Limit from the right is negative infinity

Example: Determine the infinite limit: $\lim_{x \to -3^+} \frac{x+2}{x+3}$

Step 1: There is a vertical asymptote at $x = -3$, so determine if the numerator and denominator will be positive or negative for numbers approaching -3 from the right.

$$\frac{(-2.99)+2}{(-2.99)+3} = \frac{-.99}{.01} = -99$$

The numerator is negative and the denominator is positive, and $-/+ =$ negative,

$$\text{Therefore:} \quad \lim_{x \to -3^+} \frac{x+2}{x+3} = -\infty$$

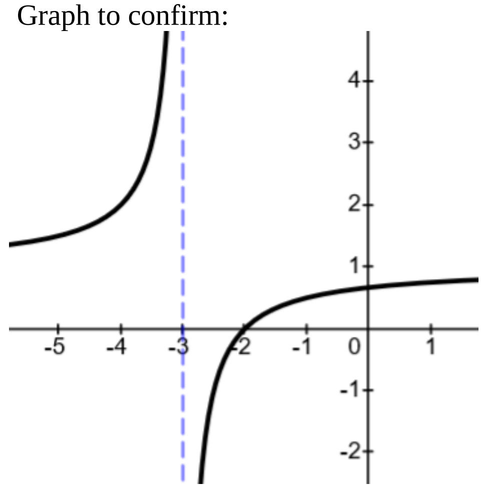
Another method is to use a table, plugging in x values approaching -3 from the right side

x	$f(x)$
-2.9	-9
-2.99	-99
-2.999	-999
-3	undefined

The function value is decreasing as x approaches -3 from the right

Therefore: $\lim_{x \to -3^+} \frac{x+2}{x+3} = -\infty$

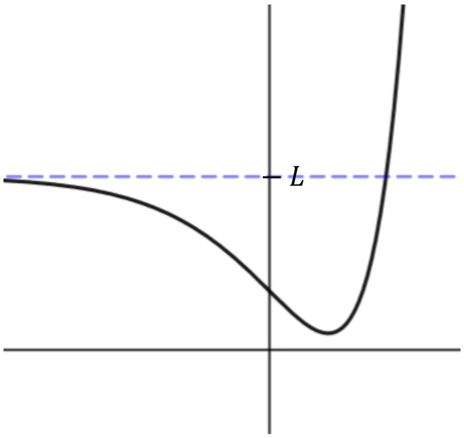
Graph to confirm:



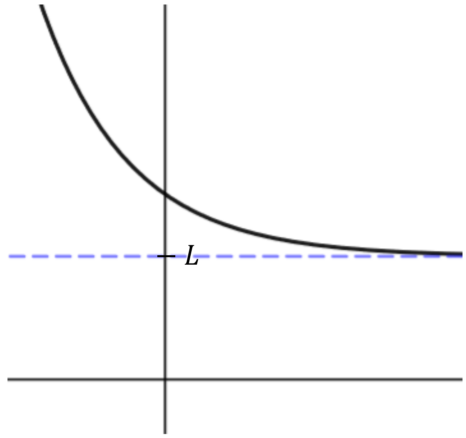
Limits at Infinity

Limits at infinity are when x goes to infinity and we determine what happens to $f(x)$.
These limits are located at horizontal asymptotes $y = L$.

$$\lim_{x\to-\infty} f(x) = L$$



$$\lim_{x\to\infty} f(x) = L$$



Finding Limits at Infinity for rational functions:

- If degree of the numerator is less than the degree of the denominator, the limit is zero.
- If the degree of the numerator is the same as the degree of the denominator, the limit is the ratio of the leading coefficients.
- If the degree of the numerator is larger than the degree of the denominator, the limit goes to infinity or negative infinity.

Examples:

$$\lim_{x\to\infty} \frac{6x^3+4}{5x^2-3x}$$ Degree is higher on top, so the limit goes to infinity.

$$\lim_{x\to\infty} \frac{6x^2+4}{5x^3-3x}$$ Degree is higher on the bottom, so the limit $= 0$.

$$\lim_{x\to\infty} \frac{6x^2+4}{5x^2-3x}$$ Degrees are the same on top and bottom, so the limit $= \frac{6}{5}$

Example: Using limit laws, evaluate $\lim_{x\to\infty} \frac{6x^2+4}{5x^2-3x}$

Step 1: Divide each term in the top and bottom by the highest power of x in the denominator (x^2).

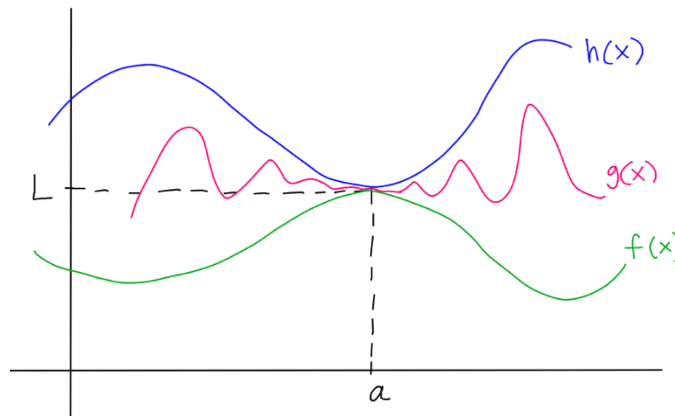
$$\lim_{x\to\infty} \frac{6x^2+4}{5x^2-3x} = \lim_{x\to\infty} \frac{\frac{6x^2}{x^2}+\frac{4}{x^2}}{\frac{5x^2}{x^2}-\frac{3x}{x^2}} = \lim_{x\to\infty} \frac{6+\frac{4}{x^2}}{5-\frac{3}{x}} = \frac{\lim_{x\to\infty}\left(6+\frac{4}{x^2}\right)}{\lim_{x\to\infty}\left(5-\frac{3}{x}\right)} = \frac{\lim_{x\to\infty} 6+\lim_{x\to\infty}\frac{4}{x^2}}{\lim_{x\to\infty} 5-\lim_{x\to\infty}\frac{3}{x}} = \frac{6+0}{5-0} = \frac{6}{5}$$

The Squeeze Theorem

If $f(x) \le g(x) \le h(x)$

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then $\lim_{x\to a} g(x) = L$



If a function g is squeezed between two other functions, f and h, if the limits of f and h are the same at the given x value, then the limit of g is also the same.

Example: Find the limit $\lim_{x\to 0} x^2 \cos\frac{1}{x}$

Step 1: Recall that the range of the cosine function is $-1 \le \cos x \le 1$

Therefore, $-1 \le \cos\frac{1}{x} \le 1$

Step 2: Multiplying this inequality by x^2 we get $-x^2 \le x^2\cos\frac{1}{x} \le x^2$

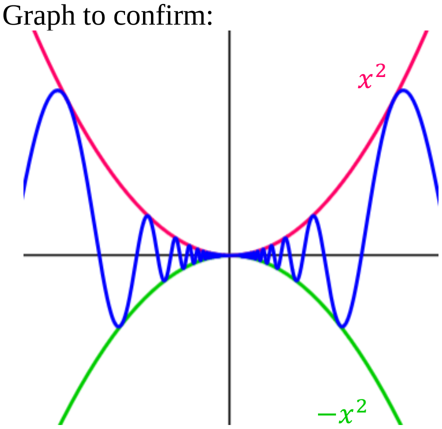
Step 3: See that the limits of $-x^2$ and x^2 are the same,

$\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$

Since $x^2\cos\frac{1}{x}$ is between $-x^2$ and x^2

So $\lim_{x\to 0} x^2\cos\frac{1}{x} = 0$

Graph to confirm:



Special Trigonometric Limits

$\lim_{x \to 0} \frac{\sin x}{x} = 1$	$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example 1: $\lim_{x \to 0} \frac{\sin(5x)}{5x} = 1$

Example 2: $\lim_{x \to 0} \frac{\sin(x)}{5x} \quad = \frac{\sin x}{x}\left(\frac{1}{5}\right) = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

Example 3: $\lim_{x \to 0} \frac{\sin(5x)}{2x} \quad = \frac{\sin(5x)}{2x}\left(\frac{5}{5}\right) = \frac{\sin(5x)}{5x}\left(\frac{5}{2}\right) = \frac{5}{2}$

Example 4: $\lim_{x \to 0} \frac{\sin x\,(1 - \cos x)}{x^2} \quad = \left(\frac{\sin x}{x}\right)\left(\frac{1 - \cos x}{x}\right) = (1)(0) = 0$

L'Hôpital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

If plugging in for x results in an indeterminate form, L'Hôpital's Rule can be used to determine the limit.

Indeterminate Forms (use L'Hôpital's Rule)	*NOT* indeterminate forms (cannot use L'Hôpital's Rule)
$\frac{0}{0}$	$\frac{0}{1} = 0$
$\frac{\infty}{\infty}$	$\frac{1}{0} = \text{undefined}$
$\infty - \infty$	$\infty + \infty = \infty$
$0 \cdot \infty$	$-\infty - \infty = -\infty$
0^0	$0^{-\infty} = \infty$
1^∞	$\infty \cdot \infty = \infty$
∞^0	

Example: Find the limit $\lim_{x \to \infty} \frac{\ln x}{x^2}$

Step 1: Plug in to see if it is indeterminant form

$$\lim_{x \to \infty} \frac{\ln \infty}{\infty^2} = \frac{\infty}{\infty}$$

Step 2: This is an indeterminant form, so use L'Hôpital's Rule

$\frac{\ln x}{x^2}$ take derivative of top and bottom $= \frac{\frac{1}{x}}{2x} = \frac{1}{x} \cdot \frac{1}{2x} = \frac{1}{2x^2}$

$$\lim_{x \to \infty} \frac{1}{2x^2} = \frac{1}{2(\infty)^2} = \frac{1}{\infty} = 0$$

Therefore:

$$\lim_{x \to \infty} \frac{\ln \sqrt{x}}{x^2} = 0$$