Absolute Maximum and Minimum

Example: Find the absolute max and min of the function on the given interval.

$f(x) = x^3 - 6x^2 + 5 \quad , \quad [-3,5]$

Step 1: Find the derivative

$$f'(x) = 3x^2 - 12x$$

Step 2: Set the derivative to zero and solve for x to get critical numbers

$$3x^2 - 12x = 0$$

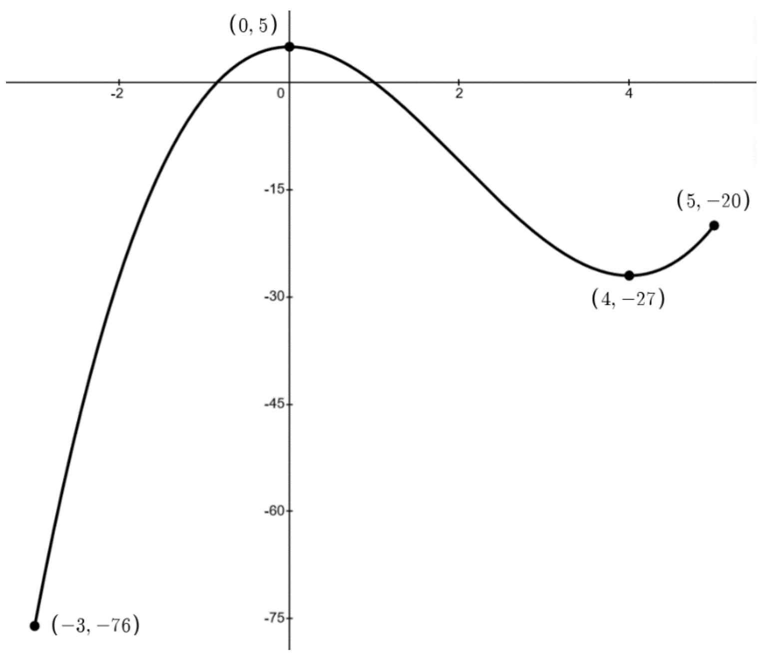
$3x(x-4) = 0$ Critical numbers are $x = 0, 4$

Step 3: Plug critical numbers and interval bounds into the original function and see which get the smallest (min) and largest (max) values.

$$f(-3) = -76, \quad f(0) = 5, \quad f(4) = -27, \quad f(5) = -20$$

Therefore, the absolute max value is $f(0) = 5$; The absolute min value is $f(-3) = -76$

Graph to verify:



First Derivative Test

Example: Find the intervals of increase and decrease, and local max and min of the following function.

$f(x) = x^3 - 6x^2 + 9x - 10$

Step 1: Find the derivative

$f'(x) = 3x^2 - 12x + 9$

Step 2: Set the derivative to zero and solve for x to get critical numbers

$3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$3(x - 3)(x - 1) = 0$ Critical numbers are $x = 1, 3$

Step 3: Use number line to test between critical numbers for increase/decrease

Plug test points into the derivative
and determine sign (positive or negative):

Test numbers: $0, 2, 4$

$f'(0) = 9$ positive

$f'(2) = -3$ negative

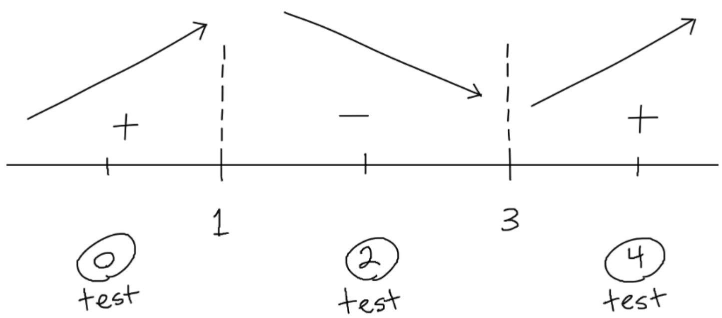
$f'(4) = 9$ positive

So intervals of increase are:

$(-\infty, 1) \cup (3, \infty)$

And intervals of decrease are:

$(1, 3)$



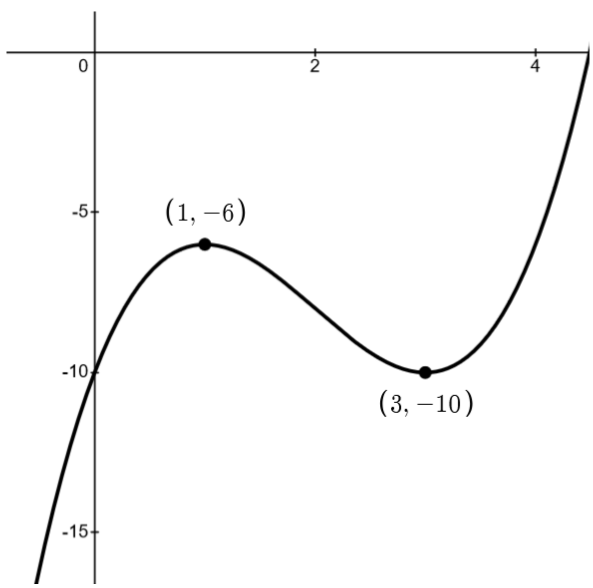
Step 4: Plug critical numbers into the original function

The number line shows there is a local max located at $x = 1$ and local min located at $x = 3$, plug in the critical values in the original function to get the values of the local max and min

$f(1) = -6, \quad f(3) = -10$

Therefore, the local max value is $f(1) = -6$ and local min value is $f(3) = -10$

Graph to verify:



Second derivative: the derivative of the 1st derivative.

Function:	Derivative:	Second Derivative:
$f(x) \rightarrow$	$f'(x) \rightarrow$	$f''(x)$

Example: Find the second derivative of the function

Function:	Derivative:	Second Derivative:
$f(x) = 2x^3$	$f'(x) = 6x^2$	$f''(x) = 12x$

Second Derivative can be used to find:

- Points of inflection
- Concavity
- Local max and min (second derivative test)
- Acceleration
- Point of diminishing returns

Concavity and Points of Inflection

Example: Find the intervals of concavity and the inflection points

$$f(x) = x^4 - 4x^3 + 2$$

Step 1: Find the 1st and 2nd derivative

$$f'(x) = 4x^3 - 12x^2 \qquad f''(x) = 12x^2 - 24x$$

Step 2: Set the 2nd derivative to zero and solve for x to find the 2nd order critical numbers

$$12x^2 - 24x = 0$$

$12x(x-2) = 0$ $\quad 12x = 0\,,\ x - 2 = 0$ $\quad$ So the 2nd order critical numbers are $x = 0\,,\ x = 2$

Step 3: Use number line to test in between critical numbers for concave up or concave down

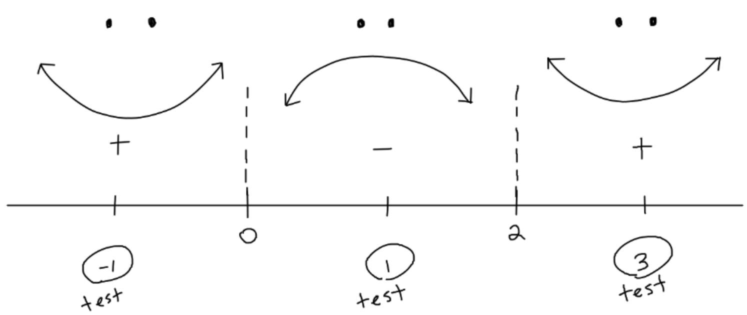
Plug test numbers into 2nd derivative and determine the sign (positive or negative)

Test numbers: $-1, 1, 3$

$f''(-1) = 36$ positive

$f''(1) = -12$ negative

$f''(3) = 36$ positive



So the function is concave up on the intervals:

$$(-\infty, 0) \cup (2, \infty)$$

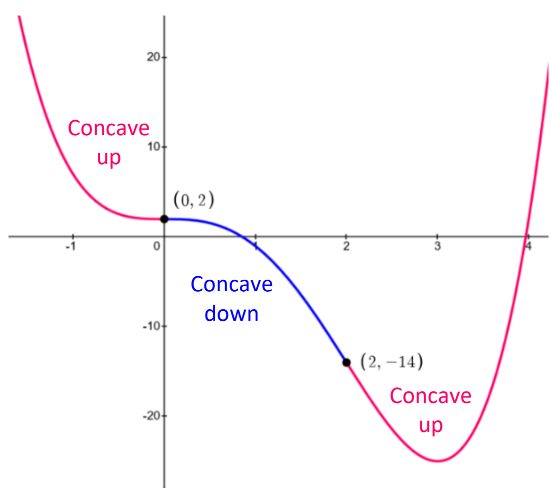
And the function is concave down on the interval:

$$(0, 2)$$

Step 4: Plug 2nd order critical numbers into original function to find inflection points

$f(0) = 2,\quad f(2) = -14$ $\quad$ Therefore, the points of inflection are $(0, 2)$ and $(2, -14)$

Graph to verify:

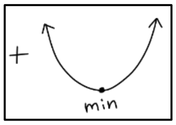


Second Derivative Test

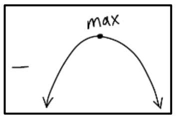
The second derivative can be used to find the local maximum and minimum of a function

Let $c =$ critical number from the 1st derivative

If $f''(c)$ is positive, then f has a local minimum at c



If $f''(c)$ is negative, then f has a local maximum at c



If $f''(c) = 0$, then the 2nd derivative test gives
no information about the critical number,
use 1st derivative test instead

Example: Find all local extrema of the function using the Second Derivative Test

$f(x) = x^3 - 3x^2 + 3$

Step 1: Find the 1st derivative and 2nd derivative

$f'(x) = 3x^2 - 6x$ $\qquad$ $f''(x) = 6x - 6$

Step 2: Set the 1st derivative to zero and solve for x to find the critical numbers

$3x^2 - 6x = 0$ $\qquad$ $3x(x-2) = 0$

$3x = 0$ $\qquad$ $x - 2 = 0$, so the critical numbers are $x = 0$, $x = 2$

Step 3: Plug the critical numbers into the 2nd derivative

$f''(0) = -6$ Negative, so a local maximum is located at $x = 0$

$f''(2) = 6$ Positive, so a local minimum is located at $x = 2$

Step 4: Plug the critical numbers into the original function to determine the values of the local max and min

$f(0) = 3$ $\qquad$ so $(0, 3)$ is a local maximum

$f(2) = -1$ $\qquad$ so $(2, -1)$ is a local minimum