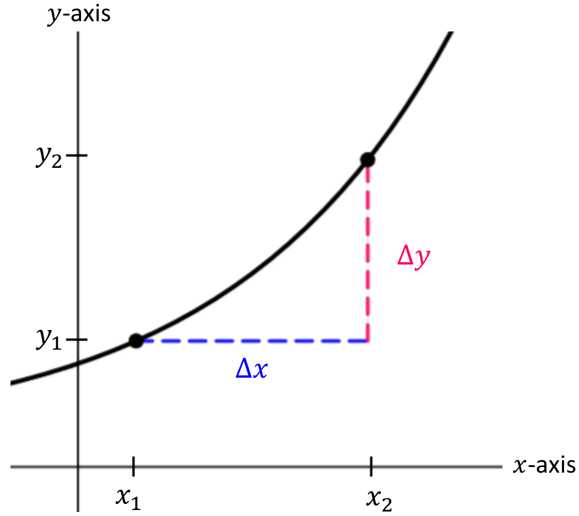
Rates of Change

A rate of change is how fast one variable (y) changes in relation to another variable (x)



The change in y (the increment of y) is $\boldsymbol{\Delta y = y_2 - y_1}$

or also written as $\boldsymbol{\Delta y = f(x_2) - f(x_1)}$

The change in x (the increment of x) is $\boldsymbol{\Delta x = x_2 - x_1}$

Equivalent Formulas (while notation can be different, these all are fundamentally the same formula)

Slope of a linear function: Average rate of change: Average Velocity: Slope of a secant line:	$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m_{\text{sec}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h}$ $\frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}$

Equivalent Formulas (these are limits of the above formulas)

Derivative: Instantaneous rate of change: Velocity: Slope of a tangent line:	$m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Definition of Derivative

The derivative is the instantaneous rate of change of a function with respect to some variable.

Definition of Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Example: Find the derivative of the function $f(x) = x^2 - 3x - 10$ using the definition of derivative.

Step 1: Find $f(x+h)$

Plug in $(x+h)$ for x in the function $f(x) = x^2 - 3x - 10$

$$f(x+h) = (x+h)^2 - 3(x+h) - 10$$

Simplify $= x^2 + 2xh + h^2 - 3x - 3h - 10$

$$f(x+h) = x^2 + 2xh + h^2 - 3x - 3h - 10$$

Step 2: Plug in $f(x+h)$ and $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 - 3x - 3h - 10) - (x^2 - 3x - 10)}{h}$$

Step 3: Combine like terms and simplify

$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{10}) - (\cancel{x^2} - \cancel{3x} - \cancel{10})}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h}$$

Factor $= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} = \lim_{h \to 0} 2x + h - 3$

Step 4: Plug zero in for h $= \lim_{h \to 0} 2x + (0) - 3 = 2x - 3$

Therefore, $f'(x) = 2x - 3$

Tangent Line

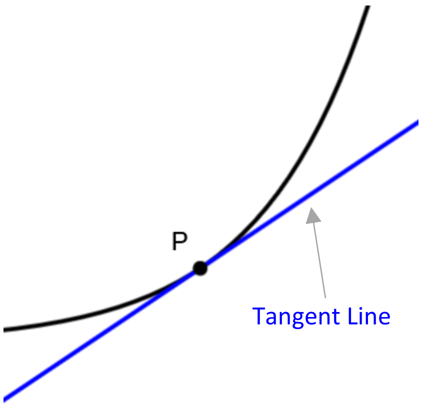
A line is tangent to a curve when it touches, but does not cross the curve at some specific point.

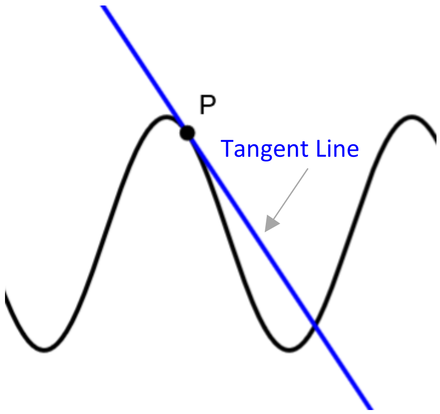
The slope of the tangent line matches the slope of the curve at the point it touches.

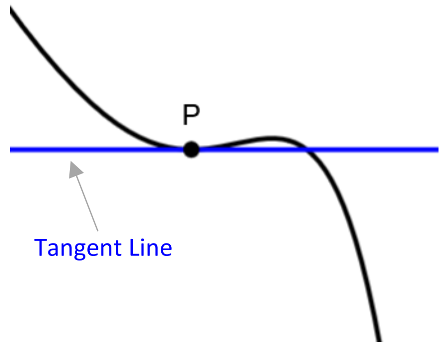
The slope of the tangent line = derivative at a specific point.

Examples of tangent lines:

These lines are all tangent to the curve at point P



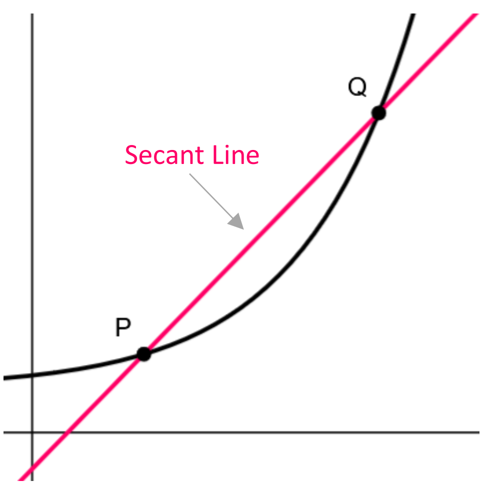


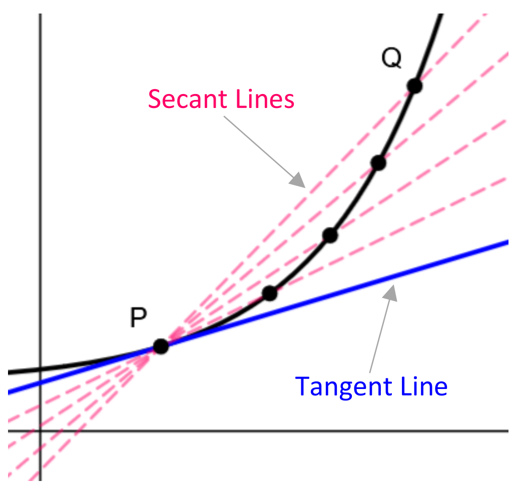


Secant Line

A secant line is a line passing through two points on a curve.

The slope of the secant line is the average rate of change of the curve from the first point to the second point. When those two points move closer together, the secant line approaches a tangent line.





As the point Q moves closer to the point P, the secant lines approach the tangent line.

Equation of the Tangent Line

Example: Find an equation of the tangent line to the curve at the given point.

$f(x) = x^2 - 3x, \quad (2, -2)$

Step 1: The derivative = the slope of the tangent line, so find the derivative to get the slope.

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2 - 3(x+h)$$

$$= x^2 + 2xh + h^2 - 3x - 3h$$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 3x - 3h) - (x^2 - 3x)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} \quad = \lim_{h \to 0} \frac{h(2x + h - 3)}{h} \quad = \lim_{h \to 0} 2x + h - 3 \quad = 2x - 3$$

So the derivative is $2x - 3$ (note: this derivative could also be found using shortcut derivative rules)

Step 2: Plug x value into derivative to get number for slope at the given point.

$f'(2) = 2(2) - 3 \quad = 1$ Therefore the slope of the tangent line is $m = 1$

Step 3: Plug the slope and given point into the point-slope equation to get a linear equation for the tangent line.

The point slope equation is $y - y_1 = m(x - x_1)$

The slope is $m = 1$ and the given point is $(2, -2)$

$y - (-2) = 1(x - 2)$

$y + 2 = x - 2$

$y = x - 4$ This is the equation of the tangent line for the given function at the point $(2, -2)$.

Graph to verify: